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Transport of a Passive Tracer by an Irregular Velocity Field

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The trajectories of a passive tracer in a turbulent flow satisfy the ordinary differential equation $\mathbf{x}'(t) = V(t, \mathbf{x}(t))$, where $V(t, x)$ is a stationary random field, the so-called Eulerian velocity. It is a nontrivial question to define the dynamics of the tracer in the case when the realizations of the Eulerian field are only spatially Hölder regular because the ordinary differential equation in question lacks then uniqueness. The most obvious approach is to regularize the dynamics, either by smoothing the velocity field (the so-called ε-regularization), or by adding a small molecular diffusivity (the so-called κ-regularization) and then pass to the appropriate limit with the respective regularization parameter. The first procedure corresponds to the Prandtl number $\mathrm{Pr} = \infty$, while the second to $\mathrm{Pr} = 0$. In the present paper we consider a two parameter family of Gaussian, Markovian time correlated fields $V(t, x)$, with the power-law spectrum. Using the infinite dimensional stochastic calculus we show the existence and uniqueness of the law of the trajectory process corresponding to a given field $V(t, x)$ for a certain regime of parameters characterizing the spectrum of the field. Additionally, this law is the limit, in the sense of the weak convergence of measures, of the laws obtained as a result of any of the described regularizations. The so-called Kolmogorov point, that corresponds to the parameters characterizing the relaxation time and energy spectrum of a turbulent, three dimensional flow, belongs to the boundary of the parameter regime considered in the paper.

KEY WORDS: Tracer dynamics; quasi-Lagrangian canonical process.

1. INTRODUCTION

The passive tracer model is used in statistical hydrodynamics to describe transport of mass in a turbulent medium. It has, on the one hand, a very

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simple formulation, on the other hand, it poses quite a challenge for a rigorous mathematical analysis. In the single particle case the trajectory of a passive tracer is a solution of an ordinary differential equation

$$\frac{d\mathbf{x}(t)}{dt} = V(t, \mathbf{x}(t)), \qquad \mathbf{x}(0) = 0, \tag{1.1}$$

where $V(t, x)$, $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, the so-called *Eulerian velocity*, is a d-dimensional time stationary, spatially homogeneous random field.

In the above framework one implicitly assumes that the trajectory of the process is indeed determined by (1.1), which requires an existence and uniqueness result for the solutions of the equation, guaranteed, for example, by the Lipschitz spatial regularity of $V(t, x)$. On the other hand, it has been long argued in the classical turbulence theory, see, e.g., ref. 1, that in dimension 3 the Eulerian velocity field would have only Hölder regular realizations (with Hölder exponent less than $1/3$) in the limit of infinite Reynolds numbers. However, real flows are always regularized at small scales due to the presence of viscous effects. To describe this type of regularization one can introduce the random field $V_\varepsilon(t, x)$ obtained from $V(t, x)$ by truncating the Fourier modes that correspond to the wave numbers bigger than a certain cut-off level, say $1/\varepsilon$ (the so-called ultraviolet cut-off). Then $V_\varepsilon(t, x)$ is analytic in the spatial variable. One can therefore define unambiguously $\mathbf{x}_\varepsilon(t)$ as the solutions of the ordinary differential equation

$$\frac{d\mathbf{x}_\varepsilon(t)}{dt} = V_\varepsilon(t, \mathbf{x}_\varepsilon(t)), \qquad \mathbf{x}_\varepsilon(0) = 0. \tag{1.2}$$

The multidimensional statistics of the trajectory can be then described by the probability density functions (PDF)

$$P_\varepsilon(x_1,..., x_N; t_1,..., t_N) = \left\langle \prod_{i=1}^{N} \delta(x_i - \mathbf{x}_\varepsilon(t_i)) \right\rangle \tag{1.3}$$

for all $t_1,..., t_N \in \mathbb{R}$. Here $\langle \cdot \rangle$ denotes the ensemble average over the realizations of the Eulerian flow. The PDF that corresponds to (1.1) is given by

$$P(x_1,..., x_N; t_1,..., t_N) := \lim_{\varepsilon \to 0+} P_\varepsilon(x_1,..., x_N; t_1,..., t_N) \tag{1.4}$$

if the limit on the right hand side of (1.4) exists. The limit of PDF-s could be understood, e.g., in the weak sense. The existence of the limit in question is far from being obvious because, as we have already mentioned, in the case of highly turbulent flow the right hand side of (1.1) does not satisfy the classical uniqueness hypotheses. On the other hand, the property described by (1.4) is quite fundamental if one wishes to define mathematically sound model of turbulent transport in the regime of very large Reynolds numbers.

Obviously, the regularization considered in (1.2) is not the only possible. Another regularizing effect comes from the molecular diffusivity. One can consider the solution of the Itô stochastic differential equation

$$d\mathbf{x}^{\kappa}(t)=V(t,\mathbf{x}^{\kappa}(t))\,dt+\sqrt{2\kappa}\,d\beta(t),\qquad \mathbf{x}^{\kappa}(0)=0, \tag{1.5}$$

where $\kappa>0$ is a (usually small) molecular diffusivity and $\beta(t)$ is a d-dimensional standard Brownian motion independent of $V(t,x)$. Then, due to the well known result of Veretennikov, see ref. 2, $\mathbf{x}^{\kappa}(t)$ is unambiguously defined as a strong solution to (1.5), even for Hölder continuous drift $V(t,x)$ and in analogy with (1.3) and (1.4) one can define

$$\bar{P}(x_1,\dots,x_N;t_1,\dots,t_N):=\lim_{\kappa\to 0+}P^{\kappa}(x_1,\dots,x_N;t_1,\dots,t_N) \tag{1.6}$$

where

$$P^{\kappa}(x_1,\dots,x_N;t_1,\dots,t_N)=\left\langle \mathbb{E}^{\beta}\left[\prod_{i=1}^{N}\delta(x_i-\mathbf{x}^{\kappa}(t_i))\right]\right\rangle \tag{1.7}$$

and $\mathbb{E}^{\beta}$ is the expectation corresponding to averaging over the realizations of the Brownian paths $\beta(t)$. If we define *the Prandtl number* as $\mathrm{Pr}:=\varepsilon/\kappa$ the limit of (1.4) corresponds to $\mathrm{Pr}=0$, while the one of (1.6) to $\mathrm{Pr}=\infty$. Besides of the fundamental question whether the limits of (1.3) and (1.6) exist or not one can inquire also whether the PDF-s obtained by means of both limiting procedures are identical, i.e.,

$$\bar{P}(x_1,\dots,x_N;t_1,\dots,t_N)=P(x_1,\dots,x_N;t_1,\dots,t_N) \tag{1.8}$$

for all $x_1,\dots,x_N\in\mathbb{R}^d$, $t_1,\dots,t_N\in\mathbb{R}$. If, the equality in (1.8) does not hold one would have to complement the statement of the problem on determining the statistics of a solution to (1.1) from the law of the right hand side $V(t,x)$ by specifying the type of the regularization involved.

In the present paper we show the existence of the limits in question and the equality (1.8) for a family of time stationary, spatially homogeneous zero mean Ornstein–Uhlenbeck, i.e., Gaussian and Markovian, fields $V(t,x)$. The detailed description of such a family is given in formulas (1.9)–(1.11) later. In fact the existence of the limits in (1.4), (1.6) follows from the convergence of the laws of the respective stochastic processes claimed in Theorem 1.1 later. To show this result one needs to prove tightness of the laws of both κ and ε-regularizations (which is rather a straightforward matter) and the uniqueness of a limiting law. The latter fact follows from a stronger result, see Theorem 1.2 later, where we assert the uniqueness of the law of a trajectory process satisfying (1.1). This also implies the equality claimed in (1.8). Note that such a uniqueness result fails to be true in general. An appropriate example for the Ornstein–Uhlenbeck model considered here seems to be quite a nontrivial issue and we do not have a result to present. However, such a nonuniqueness phenomenon occurs for a two–particle process in the Kraichnan model, see refs. 3–5, where it has been shown that for suitably selected parameters (the intermediate compressibility regime of refs. 3 and 4) the ε and κ-regularization limits are different.

As we have already mentioned in the foregoing we shall consider a family of time stationary, spatially homogeneous zero mean Gaussian and Markovian fields $V(t,x)$ whose spectrum satisfies the power law, i.e., its co-variance matrix is given by

$$R(t,x) := \langle V(t,x) \otimes V(0,0) \rangle = m_1 \int_{\mathbb{R}^d} e^{\mathrm{i} x\cdot \mathbf{k}}\, e^{-m_2 |\mathbf{k}|^{2\gamma} t}\, \mathscr{E}(\mathbf{k})\, \frac{\mathrm{d}\mathbf{k}}{|\mathbf{k}|^{d-1}}, \tag{1.9}$$

with $m_1, m_2 > 0$ and $\gamma \geqslant 0$. We assume that the energy spectrum $\mathscr{E}(\cdot)$ of the field is a nonnegative, symmetric matrix valued function given by

$$\mathscr{E}(\mathbf{k}) = \begin{cases} \phi(|\mathbf{k}|) \left[B(d-1)\, \dfrac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2} + A\left(\mathbf{I} - \dfrac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}\right)\right] & \text{if} \quad d \geqslant 2; \\ \phi(|\mathbf{k}|) & \text{if} \quad d = 1. \end{cases} \tag{1.10}$$

Here ϕ: $[0,\infty) \to [0,\infty)$ is a bounded measurable function satisfying

$$\phi(r) = O(r^{1-2\alpha}) \qquad \text{as} \quad r \to \infty, \tag{1.11}$$

with a certain $\alpha > 1$. The parameter α is related to the spatial regularity of $V(t,x)$. In fact, the velocity field has almost surely spatially Hölder continuous realization with any exponent $\sigma < \alpha - 1$, see Proposition 2.2 and Remark 2.3. The parameters m_2, γ on the other hand determine the rate at which the field decorrelates in time at distances of order $1/|\mathbf{k}|$, while the parameter m_1 determines the size of the fluctuations of the field. This class of velocity fields plays an important role in statistical hydrodynamics and is often used to model turbulent transport phenomena, see, e.g., refs. 6 and 7. Of special significance are the fields whose spectrum in dimension $d \geqslant 3$

corresponds to $\gamma = 1/3$, $\alpha = 4/3$, due to the fact that for these values of parameters they satisfy the Kolmogorov–Obukhov self-similarity hypothesis for developed turbulence.

It is well known from the classical spectral representation theorem, see, e.g., ref. 8, that there exist two independent identically distributed Gaussian spectral measures $\hat{V}_0(t, \cdot)$, $\hat{V}_1(t, \cdot)$ such that

$$V(t, x) = \int_{\mathbb{R}^d} [\hat{V}_0(t, d\mathbf{k}) \cos(x \cdot \mathbf{k}) + \hat{V}_1(t, d\mathbf{k}) \sin(x \cdot \mathbf{k})].$$

The structure function of those measures is given by the relation

$$\langle \hat{V}_i(t, d\mathbf{k}) \otimes \hat{V}_{i'}(s, d\mathbf{k}') \rangle = m_1 \, \delta_{i, i'} \, \delta(\mathbf{k} - \mathbf{k}') \, e^{-m_2 \, |\mathbf{k}|^{2\gamma} \, |t-s|} \mathscr{E}(\mathbf{k}) \, \frac{d\mathbf{k} \, d\mathbf{k}'}{|\mathbf{k}|^{d-1}},$$

with $i, i' \in \{0, 1\}$ and the regularization of the field due to viscous effects can be then mimicked by the field

$$V_\varepsilon(t, x) = \int_{|\mathbf{k}| \leqslant 1/\varepsilon} [\hat{V}_0(t, d\mathbf{k}) \cos(x \cdot \mathbf{k}) + \hat{V}_1(t, d\mathbf{k}) \sin(x \cdot \mathbf{k})]. \tag{1.12}$$

Our main results, that follow directly from a bit more abstractly formulated Theorem 2.5 of Section 2, are stated in Theorems 1.1 and 1.2 below.

Theorem 1.1. For any $\varepsilon > 0$ let $\mathscr{L}_\varepsilon$ be the joint law, in an appropriate path space, of the random element $(V_\varepsilon, \mathbf{x}_\varepsilon)$, i.e., the pair consisting of the ε-regularization of the drift, given by (1.12), and the uniquely determined solution of (1.2). Suppose that the parameters γ, α satisfy

$$2\gamma + \alpha > 2. \tag{1.13}$$

Then, the laws $\mathscr{L}_\varepsilon$ converge weakly, as $\varepsilon \to 0$, to a certain probability measure $\mathscr{L}$.

An analogous statement to the one formulated above holds also for the convergence of the laws of κ-regularizations described by (1.5).

An obvious consequence of the definition of weak convergence is that for any random element $(V, \mathbf{x})$ whose law coincides with $\mathscr{L}$ the process $(\mathbf{x}(t))_{t \geqslant 0}$ satisfies (1.1), a.s. over the respective probability space. The joint limiting law of the trajectory process and the Eulerian velocity field is, in some sense, unique. Namely, we have the following.

Theorem 1.2. Suppose that γ, α satisfy (1.13) and the Gaussian, Markovian random field $V(t, x)$ is defined by (1.9)–(1.11). Assume also that $\mathbf{x}(t)$ satisfies (1.1) and is non-anticipative w.r.t. the natural filtration corresponding to $V(t, \cdot)$, $t \geqslant 0$. Then, the law of the random element $(V, \mathbf{x})$, coincides with $\mathscr{L}$.

An important physical ramification of Theorem 1.2 is the fact that in the regime of large Reynolds numbers and for the values of parameters γ and α satisfying (1.13) the statistics of the sample trajectory of a tracer shall not depend on the type of the regularizing effects involved. In particular, the limiting laws corresponding to both κ and ε-regularization procedures must coincide. Additionally, we note that the Kolmogorov point $K = (4/3, 1/3)$ lies precisely on the boundary of the region on the (α, γ)-plane that is determined by condition (1.13). We observe also that the aforementioned condition determines the region of parameters for the validity of the Gronwall inequality, that is crucial in our argument on the uniqueness of the limiting law, see (4.5) and (4.6) later. This fact could potentially indicate that the uniqueness of the law for the solution to (1.1) might fail to be true if $\alpha + 2\gamma < 2$. However, we do not have any rigorous result to support this conjecture.

As we mentioned earlier Theorems 1.1 and 1.2 follow from Theorem 2.5 of Section 2 later. The proof of the latter is based on the observation that the quasi-Lagrangian processes $\eta_\varepsilon(t, \cdot) := V_\varepsilon(t, \mathbf{x}_\varepsilon(t) + \cdot)$, $\eta^\kappa(t, \cdot) := V^\kappa(t, \mathbf{x}^\kappa(t) + \cdot)$, $t \geqslant 0$ are stochastic processes that satisfy, in a certain sense, some Itô stochastic partial differential equations. We show in Section 4 that the solutions of those equations considered over an appropriate probability space converge to a solution of a certain limiting stochastic partial differential equation. The weak convergence of the laws of the κ and ε-regularizations are the consequences of the uniqueness result on the solutions of the limiting equation, see Theorem 4.1.

At the end of this section we point out some differences between the model discussed here and the so-called *Kraichnan model*, that has been widely considered in the literature, see, e.g., refs. 9 and 10. In contrast with the situation considered here, the Kraichnan flow is a white noise (δ-correlated) in time Gaussian field with the energy spectrum described by function $\mathscr{E}(k)$ as given by (1.10) and (1.11). This corresponds to parameters $m_1 = m_2 = +\infty$ and $\gamma = 0$ in the Ornstein–Uhlenbeck flow considered here. In fact, it has been shown in refs. 11 and 12 that in the regime considered in this article and for $m_1 = m_2 = \rho^{-2}$, one can obtain the law of a particle transported by a Kraichnan flow as a limit, as $\rho \to 0+$, of the corresponding laws obtained for a particle transported under Ornstein–Uhlenbeck flows with appropriately defined ultraviolet cut-offs depending on ρ that tend to infinity, as ρ vanishes.

The principal reasons why the Kraichnan model can be treated analytically are due to its Gaussianity and decorrelation in time. In fact, the question of determining the joint law of the velocity and trajectory process (for a single particle), that is the subject of the present paper, is quite simple there with the answer independent of the limiting procedure (in particular the law of a single trajectory is that of a Brownian motion).

According to ref. 10 the flow of particles determined by a spatially irregular, white noise velocity displays a range of interesting phenomena such as branching (intrinsic stochasticity) and coalescing (shock-wave formation) of trajectories. The nature of the limiting flow depends on the one hand on the compressibility coefficient, defined as $\mathscr{P} := B/(A+B)$, cf. (1.10). On the other hand, it is also sensitive of the choice of the limiting procedure (either ε, or κ-regularization), see refs. 3 and 4 with additional clarifications contained in ref. 5. An analogous question about the description of the particle flow could be raised in the context of considered here velocity fields that display time correlations. This problem seems to be significantly harder than in the case of the Kraichnan model. We note here that certain non-rigorous results concerning this issue have been obtained in ref. 9.

2. NOTATION AND THE MAIN RESULTS

For brevity we denote by $\mathscr{S}_d$ the space of all tempered test $\mathbb{R}^d$-valued functions $\mathscr{S}(\mathbb{R}^d; \mathbb{R}^d)$. Let $\mathbb{L}^2_\rho$ be the Hilbert space consisting of all vector fields $\psi: \mathbb{R}^d \to \mathbb{R}^d$, for which the norm

$$\|\psi\|_{\mathbb{L}^2_\rho} := \int_{\mathbb{R}^d} |\psi(x)|^2 \, \vartheta_\rho(x) \, dx$$

is finite. Here $\vartheta_\rho(x) := (1+|x|^2)^{-\rho}$, $x \in \mathbb{R}^d$, and $\rho \geqslant 0$. We shall also denote by $\langle \cdot, \cdot \rangle_{\mathbb{L}^2_\rho}$ the scalar product corresponding to the norm $\|\cdot\|_{\mathbb{L}^2_\rho}$. Next given $\rho \in [0, \infty)$ and $\sigma \in (0, 1)$ we write

$$\mathscr{C}_\rho := \{\psi \in C(\mathbb{R}^d; \mathbb{R}^d) : \|\psi\|_{\mathscr{C}_\rho} := \sup_{x \in \mathbb{R}^d} |\psi(x)|_{\mathbb{R}^d} \, \vartheta_\rho(x) < \infty\},$$

$$\mathscr{C}^\sigma_\rho := \left\{\psi \in \mathscr{C}_\rho : [\psi]_{\mathscr{C}^\sigma_\rho} = \sup_{x,\, y \in \mathbb{R}^d,\, x \neq y} \frac{|\psi(x) \, \vartheta_\rho(x) - \psi(y) \, \vartheta_\rho(y)|}{|x-y|^\sigma} < \infty\right\}.$$

We equip $\mathscr{C}^\sigma_\rho$ with the norm $\|\cdot\|_{\mathscr{C}^\sigma_\rho} := \|\cdot\|_{\mathscr{C}_\rho} + [\cdot]_{\mathscr{C}^\sigma_\rho}$ and set $\mathscr{C}^0_\rho := \mathscr{C}_\rho$.

Remark 2.1. Note that $\mathscr{S}_d$ is not dense in $\mathscr{C}_\rho$ in $\|\cdot\|_{\mathscr{C}_\rho}$-norm. However, it is dense in any $\|\cdot\|_{\mathscr{C}_\theta}$-norm with $\theta > \rho$. Note that $\mathscr{C}_\theta \subset \mathbb{L}^2_\rho$ (is continuously embedded) provided that $\rho - 2\theta > d/2$.

We assume that $V(\cdot)$ is an infinite-dimensional Ornstein–Uhlenbeck process being a stationary solution to the following stochastic equation

$$\mathrm{d}V(t) = -AV(t)\,\mathrm{d}t + B\,\mathrm{d}W(t). \tag{2.1}$$

Here W is a cylindrical Wiener process on $\mathbb{L}^2 := L^2(\mathbb{R}^d;\, \mathbb{R}^d)$ defined over a filtered probability space $\mathfrak{A} = (\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geqslant 0}, \mathbb{P})$, see, e.g., ref. 13. We shall denote the expectation w.r.t. probability $\mathbb{P}$ by $\langle\cdot\rangle$. A and B are pseudo-differential operators

$$\widehat{A\psi}(\mathbf{k}) = m_2\,|\mathbf{k}|^{2\gamma}\,\hat{\psi}(\mathbf{k}),\ \ \widehat{B\psi}(\mathbf{k}) = \sqrt{2m_1m_2}\,|\mathbf{k}|^{\gamma+(1-d)/2}\,\mathscr{E}^{1/2}(\mathbf{k})\,\hat{\psi}(\mathbf{k}), \quad \mathbf{k}\in\mathbb{R}^d,$$

with $\gamma \geqslant 0$. Notice that the operator $-A$ with the domain

$$D(A) = \left\{\psi\in\mathbb{L}^2 : \int_{\mathbb{R}^d} |\hat{\psi}(\mathbf{k})|^2\,|\mathbf{k}|^{4\gamma}\,\mathrm{d}\mathbf{k} < \infty\right\}$$

generates a C_0-semigroup of self-adjoint operators $(S(t))_{t\geqslant 0}$ on $\mathbb{L}^2$. Process $W(\cdot)$ does not live in $\mathbb{L}^2$. It takes values in any Hilbert space H such that $\mathbb{L}^2 \subset H$ with a Hilbert–Schmidt imbedding. Given $\gamma \geqslant 0$ we set $\Theta(\gamma) = \gamma$ if $\gamma \notin \mathbb{Z}$ and $+\infty$ otherwise. Assume that $\rho \in (d/2, d/2 + \Theta(\gamma))$. Then, see part (ii) of Proposition 2,[14] $(S(t))_{t\geqslant 0}$ has a unique extension, that we denote by the same symbol, to a C_0-semigroup on $\mathbb{L}^2_\rho$. Moreover, see ref. 14, Appendix A, given $t > 0$, $S(t)\,B$ extends to a Hilbert–Schmidt operator from $\mathbb{L}^2$ to $\mathbb{L}^2_\rho$. Thus, we can treat (2.1) as a stochastic evolution equation in $\mathbb{L}^2_\rho$-space, see, e.g., ref. 13. The following proposition gathers results proven already in ref. 14, Sections 2.2 and 2.3.

Proposition 2.2. Let $\rho \in (d/2, d/2 + \Theta(\gamma))$. Suppose that ζ is an $\mathfrak{F}_0$-measurable, $\mathbb{L}^2_\rho$-valued, of zero mean, normally distributed random element with spatially homogeneous law, whose co-variance operator Σ is given by

$$\widehat{\Sigma\psi}(\mathbf{k}) := \sqrt{m_1}\,|\mathbf{k}|^{(1-d)/2}\,\mathscr{E}^{1/2}(\mathbf{k})\,\hat{\psi}(\mathbf{k}), \qquad \mathbf{k}\in\mathbb{R}^d, \quad \psi\in\mathscr{S}_d.$$

Then, there is a unique solution $V(\cdot)$ to (2.1) satisfying $V(0) = \zeta$. It defines a space-time stationary Gaussian random field with jointly continuous realizations and the co-variance matrix given by (1.9) and (1.10). Moreover, for an arbitrary $\theta > 0$ and $0 \leqslant \sigma < \alpha - 1$, $V(\cdot)$ has almost surely continuous C^σ_θ-valued trajectories.

Remark 2.3. Let $V(\cdot)$ be a stationary solution and let ϕ be a function introduced in (1.10). Assume that $\alpha > 1$ is such that the estimate (1.11) is sharp, that is there are $R < \infty$ and $\phi_0 > 0$ such that $\phi(r)\, r^{2\alpha-1} \geqslant \phi_0$ for $r \geqslant R$. Then, it is easy to see that there is a constant $c > 0$ such that the co-variance matrix $R(t, x)$ of the respective Gaussian random field $V(t, x)$, $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ (cf. (1.9)) satisfies

$$\forall x\colon |x| \leqslant 1, \qquad \frac{1}{c}\,|x|^{2\alpha-2} \leqslant |R(t, x) - R(t, 0)| \leqslant c\,|x|^{2\alpha-2}. \tag{2.2}$$

Assume that almost all realizations of $V(t, \cdot)$, $t \geqslant 0$ are spatially Hölder continuous with an exponent σ. As a result of (2.2) we have necessarily, $\sigma \in (0, \alpha - 1)$. In particular, if $\alpha < 2$, then $V(\cdot)$ does not have Lipschitz continuous realizations.

Definition 2.4. By *a weak solution to* (1.1) we mean any triple $(\mathfrak{A}, V(\cdot), \mathbf{x}(\cdot))$ consisting of a filtered probability space $\mathfrak{A}$, a stationary solution $V(\cdot)$ to (2.1) driven by an $(\mathfrak{F}_t)$-adapted cylindrical Wiener process over $\mathfrak{A}$, and a measurable $(\mathfrak{F}_t)$-adapted process $\mathbf{x}\colon \Omega \times [0, +\infty) \to \mathbb{R}^d$ such that for any $t \geqslant 0$,

$$\mathbf{x}(t) = \int_0^t V(s, \mathbf{x}(s))\, ds, \qquad \mathbb{P}\text{-a.s.} \tag{2.3}$$

Let $T > 0$, and let ξ be any stochastic process with continuous trajectories in $\mathbb{R}^d$ or $C(\mathbb{R}^d; \mathbb{R}^d)$. By $\mathscr{L}_T(\xi)$ we denote its law in the appropriate space $C([0, T]; \mathbb{R}^d)$, or $C([0, T]; C(\mathbb{R}^d; \mathbb{R}^d))$. Finally $\mathscr{L}(\xi)$ shall denote its law in the space corresponding to the infinite time interval.

From the Veretennikov results on the existence and uniqueness of a solution to stochastic differential equations with a nondegenerate drift, see, e.g., ref. 2 one can easily deduce the existence and uniqueness of a strong solution $\mathbf{x}^\kappa(\cdot)$ to (1.5) with any $\kappa > 0$. These solutions shall be also referred to as *the κ-regularization of the trajectory process.*

The ε-regularization procedure starts with the regularization of the Eulerian velocity field. Let ζ be as in Proposition 2.2, i.e., a zero mean, spatially homogeneous, Gaussian random element. There exist then, see ref. 8, two $\mathbb{R}^d$-valued, independent, Gaussian spectral random measures $\hat{\zeta}_0$, $\hat{\zeta}_1$ defined over $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ such that

$$\zeta(x) = \int_{\mathbb{R}^d} [\cos(\mathbf{k} \cdot x)\, \hat{\zeta}_0(d\mathbf{k}) + \sin(\mathbf{k} \cdot x)\, \hat{\zeta}_1(d\mathbf{k})], \qquad x \in \mathbb{R}^d. \tag{2.4}$$

Let $\rho \in (d/2, d/2+\Theta(\gamma))$. For any $\varepsilon > 0$ the ε-regularization of $V(\cdot)$, denoted by $V_\varepsilon(\cdot)$, is the stationary solution of

$$\mathrm{d}V(t) = -AV(t)\,\mathrm{d}t + B_\varepsilon\,\mathrm{d}W(t), \tag{2.5}$$

where

$$\widehat{B_\varepsilon\psi}(\mathbf{k}) = \sqrt{2m_1m_2}\,\mathbf{1}_{[|\mathbf{k}|\leqslant 1/\varepsilon]}(\mathbf{k})\,|\mathbf{k}|^{\gamma+(1-d)/2}\,\mathscr{E}^{1/2}(\mathbf{k})\,\hat{\psi}(\mathbf{k}), \qquad \mathbf{k}\in\mathbb{R}^d,$$

and the initial value $V_\varepsilon(0) = \zeta_\varepsilon$ is an $\mathfrak{F}_0$-measurable Gaussian, $\mathbb{L}^2_\rho$-valued random element given by

$$\zeta_\varepsilon(x) = \int_{|\mathbf{k}|\leqslant 1/\varepsilon} [\cos(x\cdot\mathbf{k})\,\hat{\zeta}_0(\mathrm{d}\mathbf{k}) + \sin(x\cdot\mathbf{k})\,\hat{\zeta}_1(\mathrm{d}\mathbf{k})],$$

with $\hat{\zeta}_0$, $\hat{\zeta}_1$ spectral measures appearing in (2.4). Note that the random field $\zeta_\varepsilon(\cdot)$ is of zero mean with the co-variance operator given by

$$\widehat{\Sigma_\varepsilon\psi}(\mathbf{k}) := \sqrt{m_1}\,\mathbf{1}_{[|\mathbf{k}|\leqslant 1/\varepsilon]}(\mathbf{k})\,|\mathbf{k}|^{(1-d)/2}\,\mathscr{E}^{1/2}(\mathbf{k})\,\hat{\psi}(\mathbf{k}), \qquad \mathbf{k}\in\mathbb{R}^d, \quad \psi\in\mathscr{S}_d.$$

By virtue of Proposition 2.2, V_ε determines a random field that is space-time continuous, Lipschitz regular in the space variable (in fact even real analytic), and with a sub-linear growth. Therefore, for any $\varepsilon > 0$, (1.2) has a unique measurable adapted solution $\mathbf{x}_\varepsilon(\cdot)$, called *the ε-regularization of the trajectory process.*

The main result of the present paper is the following theorem on the existence and uniqueness of a weak solution.

Theorem 2.5. Assume that $\alpha+2\gamma > 2$. Then, there exists a weak solution $(\mathfrak{A}, V(\cdot), \mathbf{x}(\cdot))$ to (1.1). For arbitrary two weak solutions $(\mathfrak{A}, V(\cdot), \mathbf{x}(\cdot))$ and $(\bar{\mathfrak{A}}, \bar{V}(\cdot), \bar{\mathbf{x}}(\cdot))$ one has $\mathscr{L}(V, \mathbf{x}) = \mathscr{L}(\bar{V}, \bar{\mathbf{x}})$. Moreover, we have also the convergence of the joint laws $\mathscr{L}(V, \mathbf{x}^\kappa) \Rightarrow \mathscr{L}(V, \mathbf{x})$, as $\kappa \to 0$, and $\mathscr{L}(V_\varepsilon, \mathbf{x}_\varepsilon) \Rightarrow \mathscr{L}(V, \mathbf{x})$, as $\varepsilon \to 0$.

We conclude this section with a series of remarks which pertain to certain questions related to the notion of weak solution introduced in Definition 2.4.

Remark 2.6. Let $(\mathfrak{A}, V(\cdot), \mathbf{x}(\cdot))$ be the weak solution in the sense of Definition 2.4. By $\mathfrak{V}$ we denote the σ-algebra of events generated by $V(\cdot)$. The question is whether the process $\mathbf{x}(\cdot)$ is $\mathfrak{V}$-measurable. If so, then the law of $\mathbf{x}(\cdot)$ conditioned on $\mathfrak{V}$ is a trivial measure for $\mathbb{P}$-a.s. realization of $V(\cdot)$. If this is not the case we say that the solutions to (1.1) have the property of *intrinsic stochasticity*. We do not know whether in the regime considered in Theorem 2.5 the solutions have this property or not. The answer seems to be beyond the scope of our present approach, based on the analysis of the regularized a trajectory process corresponding to the motion of a single particle. What is likely needed is some insight into the behavior of at least two particle system. The obstacle to our approach in that case is the fact that the corresponding two particle analogue of the quasi–Lagrangian process introduced in Section 3 is not Markovian and we cannot apply then tools of the infinite dimensional stochastic calculus.

Remark 2.7. Below we formulate a condition that guarantees that the trajectory process is deterministic, when conditioned on the information about the drift.

Definition 2.8. We say that the solutions of (1.1) satisfy *the pathwise uniqueness condition* (PUC) if, given a filtered probability space $\mathfrak{A}$, a stationary solution $V(\cdot)$ to (2.1) driven by an $(\mathfrak{F}_t)$-adapted cylindrical Wiener process over $\mathfrak{A}$, there exists at most one, up to stochastic equivalence, $(\mathfrak{F}_t)$-adapted process $\mathbf{x}\colon \Omega\times[0,+\infty)\to\mathbb{R}^d$ such that (2.3) holds.

Indeed, assume that PUC does not hold. Then one could find a filtered probability space $\mathfrak{A}$, a stationary solution $V(\cdot)$ and two nonequivalent processes $\mathbf{x}(\cdot)$, $\bar{\mathbf{x}}(\cdot)$ satisfying (2.3). Using these processes one can define a weak solution $(\tilde{\mathfrak{A}}, \tilde{V}(\cdot), \tilde{\mathbf{x}}(\cdot))$ that is intrinsically stochastic. Let ν_p be the Bernoulli measure on $\{0,1\}$ with the success probability $p\in(0,1)$. Define $\tilde{\mathfrak{A}}$ as the filtered probability space corresponding to $\tilde{\Omega}:=\{0,1\}\times\Omega$, $\tilde{\mathbb{P}}:=\nu_p\otimes\mathbb{P}$, $(\tilde{\mathfrak{F}}_t)$ is the obvious filtration related to $(\mathfrak{F}_t)$ and $\tilde{\mathfrak{F}}$ is the product σ-algebra. Set $\tilde{V}(t;\epsilon,\omega):=V(t;\omega)$, $(\epsilon,\omega)\in\tilde{\Omega}$, $\tilde{\mathbf{x}}(t;0,\omega)=\bar{\mathbf{x}}(t;\omega)$, $\tilde{\mathbf{x}}(t;1,\omega)=\mathbf{x}(t;\omega)$. It is clear that $(\tilde{\mathfrak{A}}, \tilde{V}(\cdot), \tilde{\mathbf{x}}(\cdot))$ is a weak solution, however the conditional law of $\tilde{\mathbf{x}}(\cdot)$ on $\mathfrak{V}=\sigma(\tilde{V})$ equals $p\,\delta_{\mathbf{x}(\cdot;\omega)}+(1-p)\,\delta_{\bar{\mathbf{x}}(\cdot;\omega)}$. Since $\mathbb{P}(\mathbf{x}(\cdot)\neq\bar{\mathbf{x}}(\cdot))>0$ it is not a δ-type measure.

On the other hand, suppose that PUC holds and $\omega\mapsto\mu_\omega$ is the conditional law of $\mathbf{x}(\cdot)$ onto $V(\cdot)$ for a given weak solution $(\mathfrak{A}, V(\cdot), \mathbf{x}(\cdot))$. Then $\nu_\omega:=\mu_\omega\otimes\mu_\omega$ defines a random measure on $\mathscr{X}\times\mathscr{X}$, where $\mathscr{X}:=C([0,+\infty);\mathbb{R}^d)$. Define a filtered probability space $\tilde{\mathfrak{A}}$ by $\tilde{\Omega}:=\Omega\times\mathscr{X}\times\mathscr{X}$, $\tilde{\mathfrak{F}}_t:=\mathfrak{F}_t\otimes\mathscr{C}_t\otimes\mathscr{C}_t$, $t\geqslant 0$, $\tilde{\mathbb{P}}(d\omega, d\pi_1, d\pi_2):=\mathbb{P}(d\omega)\otimes\nu_\omega(d\pi_1,d\pi_2)$ and set $\mathbf{x}(t;\omega,\pi_1,\pi_2):=\pi_1(t)$, $\bar{\mathbf{x}}(t;\omega,\pi_1,\pi_2):=\pi_2(t)$. For $\tilde{\mathbb{P}}$-a.s. realizations of (ω,π_1,π_2) both processes $\mathbf{x}(\cdot)$, $\bar{\mathbf{x}}(\cdot)$ satisfy (2.3). Hence, thanks to PUC, we have $\pi_1=\pi_2$, $\tilde{\mathbb{P}}$-a.s. but this immediately implies that μ_ω is of δ-type for $\mathbb{P}$-a.s. ω. From the above we conclude that the pathwise uniqueness condition implies triviality of the trajectory law conditioned on the information about $V(\cdot)$.

Remark 2.9. It is unclear whether the pathwise uniqueness condition of Definition 2.8 implies the uniqueness of the law of a weak solution of Definition 2.4. For that purpose we would have to strengthen PUC stating, e.g., that $\mathbf{x}(\cdot)$ considered there must be of the form $\Phi(V(\cdot))$, where $\Phi\colon C([0,+\infty);\mathscr{C}_\rho)\to\mathscr{X}$ is a certain fixed measurable functional. Such a condition could be called, after Definition 4.1.6, p. 149 of ref. 15, *the strong uniqueness condition* (SUC). It is not immediately clear whether SUC is implied by PUC, although we should mention that the respective implication holds for solutions of stochastic differential equations, see, e.g., Corollary p. 152 of ref. 15 and also Corollary 4.5 later.

3. QUASI-LAGRANGIAN PROCESS

Our approach is based on the notion of the *quasi-Lagrangian process*, which roughly speaking describes the medium from the point of view of the moving particle. Assume that $(\mathfrak{A}, V(\cdot), \mathbf{x}(\cdot))$ is a weak solution to (1.1). The corresponding quasi-Lagrangian process is defined by

$$\eta(t,x):=V(t,\mathbf{x}(t)+x),\qquad x\in\mathbb{R}^d,\quad t\geqslant 0. \tag{3.1}$$

The respective processes corresponding to the κ and ε-regularizations are given by

$$\eta^\kappa(t,x):=V(t,\mathbf{x}^\kappa(t)+x)\quad\text{and}\quad \eta_\varepsilon(t,x):=V_\varepsilon(t,\mathbf{x}_\varepsilon(t)+x),\quad x\in\mathbb{R}^d,\ t\geqslant 0.$$

Clearly, for $t\geqslant 0$ we have

$$\mathbf{x}(t)=\int_0^t \eta(s,0)\,\mathrm{d}s,\quad \mathbf{x}^\kappa(t)=\int_0^t \eta^\kappa(s,0)\,\mathrm{d}s+\sqrt{2\kappa}\,\beta(t),$$
$$\mathbf{x}_\varepsilon(t)=\int_0^t \eta_\varepsilon(s,0)\,\mathrm{d}s. \tag{3.2}$$

We derive a stochastic partial differential equation that is satisfied, in a certain sense, by the quasi-Lagrangian process η^κ, see also Theorem 1 from ref. 14. Before formulating it we need some more notation. Given ρ we write

$$\mathscr{S}(\rho):=\{\psi\in\mathscr{S}_d: 0\notin\operatorname{supp}\widehat{\psi\vartheta_\rho}\}.$$

Let $\beta(\cdot)$ be a standard d-dimensional Brownian motion given over a certain filtered probability space $\mathfrak{A}^\beta=(\Omega^\beta,\mathfrak{F}^\beta,(\mathfrak{F}^\beta_t),\mathbb{P}^\beta)$. We let $\mathfrak{A}\otimes\mathfrak{A}^\beta$

be the filtered probability space $(\Omega\times\Omega^{\beta}, \mathfrak{A}\otimes\mathfrak{A}^{\beta}, (\mathfrak{F}_t\otimes\mathfrak{F}_t^{\beta}), \mathbb{P}\otimes\mathbb{P}^{\beta})$. Furthermore, denote by $\langle\cdot,\cdot\rangle$ the scalar product on $\mathbb{L}^2$, or its extension to the bilinear form on $\mathscr{S}'_d\times\mathscr{S}_d$.

Theorem 3.1. Let $\kappa\geqslant 0$, $\rho\in(d/2, d/2+\Theta(\gamma))$, $\mathfrak{A}$ be a filtered probability space. Assume further that $W(\cdot)$ is an adapted cylindrical Wiener process and $V_X(\cdot)$ is a solution to (2.1) with an $\mathfrak{F}_0$-measurable initial condition X being a random element in $\mathbb{L}^2_\rho$. Suppose that $\mathbf{y}\colon \Omega\times\Omega^{\beta}\times[0,+\infty)\to\mathbb{R}^d$ is measurable, $(\mathfrak{F}_t\otimes\mathfrak{F}_t^{\beta})$-adapted, continuous in t, and such that for $\mathbb{P}$-a.s. $\omega\in\Omega$ it satisfies

$$\mathrm{d}\mathbf{y}(t)=V_X(t,\mathbf{y}(t))\,\mathrm{d}t+\sqrt{2\kappa}\,\mathrm{d}\beta(t),\qquad \mathbf{y}(0)=0. \tag{3.3}$$

Set $\eta_X(t,x)=V_X(t,\mathbf{y}(t)+x)$, $x\in\mathbb{R}^d$, $t\geqslant 0$.

Then, there is a cylindrical Wiener process $W(\cdot)$ on $\mathfrak{A}\otimes\mathfrak{A}^{\beta}$, independent of $\beta(\cdot)$ such that for all $\psi\in\mathscr{S}(\rho)$ and $t\geqslant 0$, $\mathbb{P}\otimes\mathbb{P}^{\beta}$-a.s.,

$$\begin{aligned}\langle\eta_X(t),\psi\rangle_{\mathbb{L}^2_\rho}&=\langle X,\psi\rangle_{\mathbb{L}^2_\rho}\\ &\quad+\int_0^t[\langle\eta_X(s),\vartheta_{-\rho}(-A+\kappa\Delta)(\vartheta_\rho\psi)-\vartheta_{-\rho}\eta_X(s,0)\cdot\nabla(\vartheta_\rho\psi)\rangle_{\mathbb{L}^2_\rho}]\,\mathrm{d}s\\ &\quad+\int_0^t[\langle B\,\mathrm{d}\mathbf{W}(s),\psi\rangle_{\mathbb{L}^2_\rho}-\langle\sqrt{2\kappa}\,\vartheta_{-\rho}\,\mathrm{d}\beta(s)\cdot\nabla(\vartheta_\rho\psi),\eta_X(s)\rangle_{\mathbb{L}^2_\rho}].\end{aligned}$$

Here, given a vector $v\in\mathbb{R}^d$, the symbol $v\cdot\nabla$ denotes the directional derivative in the direction of v.

Remark 3.2. In the above theorem we assume that there is a solution to (3.3). Obviously, if $\kappa=0$ and $V_X(\cdot,\cdot)$ is not Lipschitz continuous in x the existence of a solution to (3.3) could be a nontrivial issue.

Proof of Theorem 3.1. Let $\psi\in\mathscr{S}(\rho)$. Set $\varphi=\vartheta_\rho\psi$. Then

$$\langle\eta_X(t),\psi\rangle_{\mathbb{L}^2_\rho}=\langle\eta_X(t),\varphi\rangle=\langle V_X(t,\cdot),\varphi(\cdot-\mathbf{y}(t))\rangle.$$

By Itô's formula we have

$$\begin{aligned}&\mathrm{d}\varphi(\cdot-\mathbf{y}(t))\\ &\quad=-[V_X(t,\mathbf{y}(t))\,\mathrm{d}t+\sqrt{2\kappa}\,\mathrm{d}\beta(t)]\cdot\nabla\varphi(\cdot-\mathbf{y}(t))+\kappa\Delta\varphi(\cdot-\mathbf{y}(t))\,\mathrm{d}t.\end{aligned}$$

Thus, as $W(\cdot)$ and $\beta(\cdot)$ are independent and A and B commute with the group of spatial shifts, (2.1) yields

$$\langle \eta_X(t), \psi\rangle_{\mathbb{L}^2_\rho} = \langle X, \psi\rangle_{\mathbb{L}^2_\rho} + \int_0^t \langle \eta_X(s), (-A+\kappa\Delta)\,\varphi - \eta_X(s,0)\cdot\nabla\varphi\rangle\,\mathrm{d}s$$

$$-\int_0^t \langle \sqrt{2\kappa}\,\mathrm{d}\beta(s)\cdot\nabla\varphi, \eta_X(s)\rangle + \int_0^t \langle B\,\mathrm{d}W(s), \varphi(\cdot - \mathbf{y}(s))\rangle.$$

Thus, the desired conclusion follows from the fact that

$$\langle \mathbf{W}(t), \psi\rangle := \int_0^t \langle \mathrm{d}W(s), \psi(\cdot - \mathbf{y}(s))\rangle, \qquad t\geqslant 0, \quad \psi\in\mathscr{S}(\rho) \tag{3.4}$$

defines a cylindrical Wiener process on $\mathbb{L}^2$ independent of $\beta(\cdot)$. This can be seen as follows. Since for each $\psi\in\mathscr{S}_d$, $\langle \mathbf{W}(\cdot), \psi\rangle$ is a martingale with the quadratic variation

$$\langle\!\langle \langle \mathbf{W}(\cdot), \psi\rangle, \langle \mathbf{W}(\cdot), \varphi\rangle \rangle\!\rangle_t = t\langle \psi, \varphi\rangle, \qquad t\geqslant 0, \quad \psi, \varphi\in\mathscr{S}_d,$$

$\mathbf{W}(\cdot)$ is a cylindrical Wiener process in $\mathbb{L}^2$. It is independent of $\beta(\cdot)$ as the joint quadratic variations satisfy

$$\langle\!\langle \langle \mathbf{W}(\cdot), \psi\rangle, \beta(\cdot) \rangle\!\rangle_t = 0, \qquad \forall \psi\in\mathscr{S}_d. \quad \blacksquare$$

Remark 3.3. Let $\kappa\geqslant 0$ and $\rho\in(d/2, d/2+\Theta(\gamma))$. Note that for any $v\in\mathbb{R}^d$,

$$\mathscr{S}(\rho)\subset D((-A+\kappa\Delta)^*)\cap D((v\cdot\nabla)^*).$$

In fact, see Appendix B, $\mathscr{S}(\rho)$ is a core of $(-A+\kappa\Delta)^*$ and for $\psi\in\mathscr{S}(\rho)$ one has

$$(-A+\kappa\Delta)^*\,\psi = \vartheta_{-\rho}(-A+\kappa\Delta)(\vartheta_\rho\psi),$$

$$(v\cdot\nabla)^*\,\psi = -\vartheta_{-\rho} v\cdot\nabla(\vartheta_\rho\psi).$$

Theorem 3.1 states therefore that $\eta_X(\cdot)$ can be treated as a weak solution to the following stochastic evolution differential equation

$$\frac{\partial\eta_X}{\partial t}(t,x) = (-A+\kappa\Delta)\,\eta_X(t,x) + \eta_X(t,0)\cdot\nabla\eta_X(t,x)$$

$$+B\dot{\mathbf{W}}(t,x) + \sqrt{2\kappa}\,\dot\beta(t)\cdot\nabla\eta_X(t,x), \qquad t\geqslant 0, \quad x\in\mathbb{R}^d, \tag{3.5}$$

with the initial condition $\eta_X(0) = X$. In particular, $\eta^\kappa(\cdot)$ solves (3.5), with the initial value $V(0)$.

Furthermore, if $(\mathfrak{A}, V(\cdot), \mathbf{x}(\cdot))$ is a weak solution to (1.1), then the corresponding quasi-Lagrangian process (3.1) satisfies

$$\frac{\partial \eta}{\partial t}(t, x) = -A\eta(t, x) + \eta(t, 0) \cdot \nabla \eta(t, x) + B\dot{\mathbf{W}}(t, x), \tag{3.6}$$

with the initial condition $\eta(0) = V(0)$. An analogous equation can be derived for the ε-regularization of the quasi-Lagrangian process, see ref. 14. Namely, $\eta_\varepsilon(\cdot)$ solves

$$\frac{\partial \eta_\varepsilon}{\partial t}(t, x) = -A\eta_\varepsilon(t, x) + \eta_\varepsilon(t, 0) \cdot \nabla \eta_\varepsilon(t, x) + B_\varepsilon \dot{\mathbf{W}}(t, x), \tag{3.7}$$

with $\eta_\varepsilon(0) = V_\varepsilon(0)$.

4. PATHWISE UNIQUENESS OF EQ. (3.5)

Theorem 4.1. Let $\gamma > 0$, $\kappa \geqslant 0$, and let $\sigma \in [0, 1)$ be such that $2\gamma + \sigma > 1$. Let $\rho \in (0, \gamma/2)$. Suppose that we are given filtered probability spaces $\mathfrak{A}$, $\mathfrak{A}^\beta$, a cylindrical Wiener process $W(\cdot)$ over $\mathfrak{A}$, a standard d-dimensional Brownian motion $\beta(\cdot)$ over $\mathfrak{A}^\beta$ and an $\mathfrak{F}_0$-measurable random element X in $\mathscr{C}^\sigma_\rho$. Then, there is at most one solution to (3.5) with the initial value X, in the class of all $(\mathfrak{F}_t \otimes \mathfrak{F}^\beta_t)$-adapted processes $\eta(t)$, $t \geqslant 0$ with continuous $\mathscr{C}^\sigma_\rho$-valued trajectories.

Note that for any $\rho \in (0, \gamma/2)$, there is $\theta \in (d/2, d/2 + \Theta(\gamma))$ such that $\mathscr{C}_\rho \subset \mathbb{L}^2_\theta$. In fact any $\theta \in (d/2 + 2\rho, d/2 + \Theta(\gamma)$ has the desired property.

By $\mathbf{C}_\rho$ we denote the space consisting of all continuous mappings $\psi\colon \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ such that

$$\|\psi\|_{\mathbf{C}_\rho} := \sup_{x \in \mathbb{R}} |\psi(x)|_{\mathbb{R}^d \otimes \mathbb{R}^d}\, \vartheta_\rho(x) < \infty.$$

Throughout this section we assume that ρ, κ, σ, and X are as in Theorem 4.1, and that $\eta_X(\cdot)$ and $\bar{\eta}_X(\cdot)$ are two solutions to (3.5) with continuous trajectories in $\mathscr{C}^\sigma_\rho$ starting from X. The proof of the theorem will be based on the following three lemmas.

Lemma 4.2. Let $S(\cdot)$ be the semigroup on $\mathbb{L}^2$ generated by $-A$. Then,

(i) for any $t \geqslant 0$, $S(t)$ is a bounded linear operator acting from $(\mathscr{S}_d, \|\cdot\|_{\mathscr{C}_\rho})$ into $(\mathscr{C}_\rho, \|\cdot\|_{\mathscr{C}_\rho})$. Thus, by virtue of Remark 2.1, for any $t \geqslant 0$, $S(t)$ has a unique extension to a bounded operator on $\mathscr{C}_\rho$. We denote this extension also by $S(t)$.

(ii) for any $t > 0$, $\nabla S(t)$ is a bounded linear operator from $\mathscr{C}^\sigma_\rho$ into $\mathbf{C}_\rho$. Moreover, for any $T > 0$ there is a constant $C < \infty$ such that

$$\|\nabla S(t)\|_{L(\mathscr{C}^\sigma_\rho, \mathbf{C}_\rho)} \leqslant C t^{-\frac{1-\sigma}{2\gamma}}, \qquad t \in (0, T]. \tag{4.1}$$

Due to rather technical nature of the lemma we postpone its proof till Appendix A.

Lemma 4.3. Assume that $w(\cdot)$ is an adapted process with continuous trajectories in $\mathscr{C}^\sigma_\rho$ which satisfies

$$\begin{aligned} \frac{\partial w}{\partial t}(t, x) &= (-A + \kappa\Delta)\, w(t, x) + \bar{\eta}_X(t, 0) \cdot \nabla w(t, x) \\ &\quad + \sqrt{2\kappa}\, \dot{\beta}(t) \cdot \nabla w(t, x), \qquad t \geqslant 0, \quad x \in \mathbb{R}^d, \\ w(0, x) &= 0, \qquad x \in \mathbb{R}^d. \end{aligned} \tag{4.2}$$

Then $w = 0$.

Proof. Let $w(\cdot, \cdot)$ be a solution to (4.2). Write

$$z(t, x) = w(t, x - \sqrt{2\kappa}\, \beta(t)).$$

Let ψ belong to the core $\mathscr{S}(\rho)$ of $(-A + \kappa\Delta)^*$, see Remark 3.3. Write

$$\langle z(t), \psi \rangle := \int_{\mathbb{R}^d} z(t, x) \cdot \psi(x)\, \mathrm{d}x.$$

In the same manner we define $\langle w, \psi(\cdot + \sqrt{2\kappa}\, \beta(t)) \rangle$. Then

$$\begin{aligned} \mathrm{d}\langle z(t), \psi \rangle &= \mathrm{d}\langle w(t), \psi(\cdot + \sqrt{2\kappa}\, \beta(t)) \rangle \\ &= \langle \mathrm{d}w(t), \psi(\cdot + \sqrt{2\kappa}\, \beta(t)) \rangle + \langle w(t), \mathrm{d}\psi(\cdot + \sqrt{2\kappa}\, \beta(t)) \rangle \\ &\quad + 2\kappa \langle \nabla w(t), \nabla \psi(\cdot + \sqrt{2\kappa}\, \beta(t)) \rangle\, \mathrm{d}t \\ &= \langle -Aw(t) + \eta(t, 0) \cdot \nabla w(t), \psi(\cdot + \sqrt{2\kappa}\, \beta(t)) \rangle\, \mathrm{d}t. \end{aligned}$$

Hence z is a solution to

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = -Az(t) + \bar{\eta}_X(t,0)\cdot\nabla z(t), \qquad z(0)=0.$$

Let $\theta \in (2\rho + d/2, d/2 + \Theta(\gamma))$. Note that then $\mathscr{C}_\rho$ is embedded into $\mathbb{L}^2_\theta$. Since $S(\cdot)$ is a C_0-semigroup on $\mathbb{L}^2_\theta$, and thanks to (4.1),

$$\int_0^t \|\bar{\eta}_X(s,0)\cdot\nabla[S(t-s)\,z(s)]\|_{\mathbb{L}^2_\theta}\,\mathrm{d}s$$

$$\leqslant \sup_{0\leqslant s\leqslant t} (\|\,z(s)\|_{\mathscr{C}^\sigma_\rho}\,|\bar{\eta}_X(s,0)|)\int_0^t \|\nabla S(t-s)\|_{L(\mathscr{C}^\sigma_\rho,\,\mathrm{C}_\rho)}\,\mathrm{d}s < +\infty.$$

We conclude therefore that $z(\cdot)$ is a mild solution, that is

$$z(t) = \int_0^t \bar{\eta}_X(s,0)\cdot\nabla[S(t-s)\,z(s)]\,\mathrm{d}s.$$

From this point on we are able to proceed with the argument used in the proof of Lemma 3 from ref. 14, which leads to the desired conclusion that $z(\cdot)\equiv 0$. ∎

Lemma 4.4 below was formulated not explicitly and in a little weaker form in ref. 14, see the proof of Lemma 3 from ref. 14.

Lemma 4.4. The process $v := \eta_X - \bar{\eta}_X$ satisfies

$$v(t,x) = \int_0^t v(s,0)\cdot\Psi\left(t,s,x+\int_s^t \bar{\eta}_X(r,0)\,\mathrm{d}r + \sqrt{2\kappa}\,(\beta(t)-\beta(s))\right)\mathrm{d}s \tag{4.3}$$

for all $(t,x)\in[0,+\infty)\times\mathbb{R}^d$, where

$$\Psi(t,s,x) := [\nabla S(t-s)\,\eta_X](s,x), \qquad t>s\geqslant 0, \quad x\in\mathbb{R}^d.$$

Proof. Clearly, v is a (weak in the PDE sense) solution to

$$\frac{\partial v}{\partial t}(t,x) = (-A+\kappa\Delta)\,v(t,x) + v(t,0)\cdot\nabla_X\eta(t,x)$$

$$+\bar{\eta}_X(t,0)\cdot\nabla v(t,x) + \sqrt{2\kappa}\,\dot{\beta}(t)\cdot\nabla v(t,x), \qquad t\in[0,T], \quad x\in\mathbb{R}^d,$$

with the initial condition $v(0)=0$. Let $\bar{v}$ be given by the right hand side of (4.3). We will show that $\bar{v}$ satisfies

$$\frac{\partial \bar{v}}{\partial t}(t,x)=(-A+\kappa\Delta)\,\bar{v}(t,x)+v(t,0)\cdot\nabla\eta_X(t,x)$$
$$+\bar{\eta}_X(t,0)\cdot\nabla\bar{v}(t,x)+\sqrt{2\kappa}\,\dot{\beta}(t)\cdot\nabla\bar{v}(t,x),\qquad t\in[0,T],\quad x\in\mathbb{R}^d, \tag{4.4}$$

with $\bar{v}(0)=0$. This will complete the proof of the lemma. For, the difference $w=v-\bar{v}$ satisfies (4.2) and has the desired regularity properties. Thus by Lemma 4.3, $w(\cdot)\equiv 0$ and consequently $v=\bar{v}$. In order to show (4.4) we proceed as in the proof of the previous lemma. We take a test function $\psi\in\mathscr{S}(\rho)$ and write

$$Y(t,s):=\int_s^t \bar{\eta}(r,0)\,\mathrm{d}r+\sqrt{2\kappa}\,(\beta(t)-\beta(s)).$$

Then,

$$\langle\bar{v}(t),\psi\rangle=\int_0^t \langle v(s,0)\cdot\nabla S(t-s)\,\eta_X(s),\psi(\cdot-Y(t,s))\rangle\,\mathrm{d}s$$

and using Itô's calculus one obtains

$$\mathrm{d}\langle\bar{v}(t),\psi\rangle=\langle v(t,0)\cdot\nabla\eta_X(t),\psi\rangle\,\mathrm{d}t$$
$$-\left[\int_0^t \langle v(s,0)\cdot\nabla AS(t-s)\,\eta_X(s),\psi(\cdot-Y(t,s))\rangle\,\mathrm{d}s\right]\mathrm{d}t$$
$$+\int_0^t \langle v(s,0)\cdot\nabla S(t-s)\,\eta_X(s),\mathrm{d}_t\psi(\cdot-Y(t,s))\rangle\,\mathrm{d}s'$$

$$\mathrm{d}_t\psi(\cdot-Y(t,s))=\kappa\,\Delta\psi(\cdot-Y(t,s))\,\mathrm{d}t-\bar{\eta}_X(t,0)\cdot\nabla\psi(\cdot-Y(t,s))\,\mathrm{d}t$$
$$-\sqrt{2\kappa}\,\mathrm{d}\beta(t)\cdot\nabla\psi(\cdot-Y(t,s)).$$

and (4.4) follows. ∎

Proof of Theorem 4.1. By Lemma 4.4, $v(\cdot)=\bar{\eta}_X(\cdot)-\eta_X(\cdot)$ satisfies

$$|v(t,0)|\leqslant\int_0^t |v(s,0)|\,K(t,s)\,\mathrm{d}s, \tag{4.5}$$

where

$$K(t,s)=\left|[\nabla S(t-s)\,\eta_X]\left(s,\int_s^t \bar{\eta}_X(r,0)\,\mathrm{d}r+\sqrt{2\kappa}\,(\beta(t)-\beta(s))\right)\right|.$$

Clearly, for any $f\in \mathbf{C}_\rho$ we have $|f(0)|\leqslant \|f\|_{\mathbf{C}_\rho}$. Moreover, using an elementary inequality

$$\frac{1}{1+|x-y|^2}\leqslant\frac{(1+|y|)^2}{1+|x|^2},\qquad x,y\in\mathbb{R}^d$$

we obtain

$$\|f(\cdot+y)\|_{\mathbf{C}_\rho}\leqslant(1+|y|)^{2\rho}\,\|f\|_{\mathbf{C}_\rho}.$$

Hence,

$$\begin{aligned}K(t,s)&\leqslant\left(1+\int_s^t|\bar{\eta}_X(r,0)|\,\mathrm{d}r+\sqrt{2\kappa}\,|\beta(t)-\beta(s)|\right)^{2\rho}\|\nabla S(t-s)\,\eta_X(s)\|_{\mathbf{C}_\rho}\\&\leqslant\left(1+\int_s^t|\bar{\eta}_X(r,0)|\,\mathrm{d}r+\sqrt{2\kappa}\,|\beta(t)-\beta(s)|\right)^{2\rho}\\&\quad\times\|\eta_X(s)\|_{\mathscr{C}^\sigma_\rho}\,\|\nabla S(t-s)\|_{L(\mathscr{C}^\sigma_\rho,\mathbf{C}_\rho)}.\end{aligned}$$

Taking into account the continuity of the real valued processes $\|\eta_X(\cdot)\|_{\mathscr{C}^\sigma_\rho}$, $\bar{\eta}_X(\cdot,0)$, $\beta(\cdot)$, and $v(\cdot,0)$ for any fixed $T>0$ there is a random variable C_T satisfying $\mathbb{P}(C_T<+\infty)=1$ such that

$$K(t,s)\leqslant C_T\,\|\nabla S(t-s)\|_{L(\mathscr{C}^\sigma_\rho,\mathbf{C}_\rho)},\qquad \forall 0\leqslant s<t\leqslant T,\quad \mathbb{P}\text{-a.s.},\tag{4.6}$$

Thus the desired conclusion follows from Lemma 4.2 and Gronwall's inequality. ∎

At the end of this section we formulate a corollary of Theorem 4.1 that asserts the uniqueness of the law of the solution to (3.5). Consider any Hilbert space H such that the embedding $\mathbb{L}^2$ into H is of the Hilbert–Schmidt class. Let

$$C:=C([0,\infty);\mathscr{C}^\sigma_\rho),\qquad \mathscr{W}=C([0,\infty);\mathbb{R}^d)\times C([0,\infty);H).\tag{4.7}$$

Using the Yamada–Watanabe result, see, e.g., refs. 15–17 which says that the pathwise uniqueness implies law uniqueness we obtain the following corollary to Theorem 4.1.

Corollary 4.5. Assume that γ, σ, κ, and ρ are as in Theorem 4.1. Let $\eta_X(\cdot)$, $\bar{\eta}_{\bar{X}}(\cdot)$ be two solutions to (3.5) driven by possibly different Wiener processes $(\beta(\cdot), W(\cdot))$, $(\bar{\beta}(\cdot), \bar{W}(\cdot))$ and defined on a possibly different filtered probability spaces. Assume that $\eta_X(\cdot)$, $\bar{\eta}_{\bar{X}}(\cdot)$ have continuous trajectories in $\mathscr{C}^{\sigma}_{\rho}$. Let $\mathbb{Q}$ and $\bar{\mathbb{Q}}$ be the laws of $(\eta_X(\cdot), \beta(\cdot), W(\cdot))$ and $(\bar{\eta}_{\bar{X}}(\cdot), \bar{\beta}(\cdot), \bar{W}(\cdot))$ on $\mathscr{C} \times \mathscr{W}$. If $\mathscr{L}(X) = \mathscr{L}(\bar{X})$, then $\mathbb{Q} = \bar{\mathbb{Q}}$ and, in particular we also have $\mathscr{L}(\eta_X) = \mathscr{L}(\bar{\eta}_{\bar{X}})$.

5. PROOF OF THEOREM 2.5

Proof of the Uniqueness. Let $(\mathfrak{A}, V(\cdot), \mathbf{x}(\cdot))$ and $(\bar{\mathfrak{A}}, \bar{V}(\cdot), \bar{\mathbf{x}}(\cdot))$ be any two weak solutions to (1.1), and let $\eta(t, \cdot) = V(t, \cdot + \mathbf{x}(t))$ and $\bar{\eta}(t, \cdot) = \bar{V}(t, \cdot + \bar{\mathbf{x}}(t))$, $t \geqslant 0$ be the corresponding quasi-Lagrangian environment processes. Clearly if $\gamma = 0$, then necessarily $\alpha > 2$ and consequently the Eulerian velocity field is spatially Lipschitz. Thus we need to consider only the case $\gamma > 0$.

Let us fix $\rho \in (0, \gamma/2)$. Since $2\gamma + \alpha > 2$ and $\alpha \in (1, 2)$ we can find $\sigma \in [0, \alpha - 1)$ such that $2\gamma + \sigma > 1$. Note that $\eta(\cdot)$ and $\bar{\eta}(\cdot)$ have continuous trajectories in $\mathscr{C}^{\sigma}_{\rho}$. Indeed, taking into account the definition of the quasi-Lagrangian environment process it is enough to observe that $V(\cdot)$ and $\bar{V}(\cdot)$ have continuous trajectories in $\mathscr{C}^{\sigma}_{\rho}$. This is however a direct consequence of Proposition 2.2.

Clearly all the laws $\mathscr{L}(\eta(0))$, $\mathscr{L}(V(0))$, $\mathscr{L}(\bar{V}(0))$, and $\mathscr{L}(\bar{\eta}(0))$ are identical. By Theorem 3.1, $\eta(\cdot)$ and $\bar{\eta}(\cdot)$ are solutions to (3.6) driven by possibly different Wiener processes. However, by Corollary 4.5, we have $\mathscr{L}(\eta) = \mathscr{L}(\bar{\eta})$. Taking into account (3.1) and (3.2) we obtain $\mathscr{L}(V, \mathbf{x}) = \mathscr{L}(\bar{V}, \bar{\mathbf{x}})$. ∎

Proof of the Existence and Weak Convergence of κ-Regularizations. Given $\kappa > 0$ we assume that $\mathbf{x}^{\kappa}(\cdot)$ is defined on a probability space $\mathfrak{B} = \mathfrak{A} \otimes \mathfrak{A}^{\beta}$ and satisfies (1.5) with a standard Brownian motion $\beta(\cdot)$, defined over $\mathfrak{A}^{\beta}$, and the field $V(\cdot)$, defined over $\mathfrak{A}$, corresponding to a stationary solution to (2.1) driven by a cylindrical Wiener process $W(\cdot)$. $\eta^{\kappa}(\cdot)$ is the corresponding quasi-Lagrangian process. According to Theorem 3.1 the quasi-Lagrangian process satisfies (3.5) driven by a certain cylindrical Wiener process $\mathbf{W}^{\kappa}(\cdot)$. Consider any Hilbert space H such that the embedding $\mathbb{L}^2$ into H is of the Hilbert–Schmidt class. Let $\rho \in (0, \gamma/2)$, $T > 0$ and

$$\mathscr{Z}_T = C([0, T]; \mathbb{R}^d) \times C([0, T]; \mathscr{C}_{\rho}) \times C([0, T]; \mathscr{C}_{\rho}) \\ \times C([0, T]; H) \times C([0, T]; \mathbb{R}^d).$$

We denote also by $\mathscr{X}$ the respective space defined using a semi-definite interval $[0+\infty)$. Let $\mathscr{M}^\kappa := (\mathbf{x}^\kappa(\cdot), \eta^\kappa(\cdot), V(\cdot), W(\cdot), \beta(\cdot))$ for any $\kappa \in (0, 1)$.

We will show that the family of the laws $(\mathscr{L}(\mathscr{M}^\kappa))_{\kappa \in (0,1)}$ is tight in $\mathscr{X}$. According to ref. 18 it suffices only to show that the family of the laws $(\mathscr{L}_T(\mathscr{M}^\kappa))_{\kappa \in (0,1)}$ is tight in $\mathscr{X}_T$ for any $T > 0$. To achieve that goal, it is enough to show that the laws $\mathscr{L}_T(\mathbf{x}^\kappa(\cdot), \eta^\kappa(\cdot))$, $\kappa \in (0, 1]$ are tight in

$$\tilde{\mathscr{X}}_T = C([0, T]; \mathbb{R}^d) \times C([0, T]; \mathscr{C}_\rho).$$

We can further simplify the goal taking into account the fact that $V(\cdot)$ is a random element of $C([0, T]; \mathscr{C}_\rho)$. It suffices therefore to verify that the family $\mathscr{L}_T(\mathbf{x}^\kappa(\cdot))$, $\kappa \in (0, 1]$ is tight in $C([0, T]; \mathbb{R}^d)$. Note first that, thanks to Proposition 2.2 and elementary properties of homogeneous Gaussian measures, see, e.g., Theorem 5.2 on p. 120 of ref. 19, there exist a constant $a > 0$ and a random variable C_T such that $\langle e^{aC_T^2} \rangle < \infty$ and

$$|\mathbf{x}^\kappa(t)| \leqslant \sqrt{2\kappa} \sup_{0 \leqslant t \leqslant T} |\beta(t)| + C_T \left(1 + \int_0^t |\mathbf{x}^\kappa(s)|\, \mathrm{d}s\right).$$

Hence, in particular for any $q > 0$,

$$\langle \mathbb{E}^\beta [\sup_{(\kappa, t) \in (0,1) \times [0, T]} |\mathbf{x}^\kappa(t)|^q] \rangle < \infty.$$

In consequence, for any $\epsilon > 0$ there exists a sufficiently large $M > 0$ such that

$$\langle \mathbb{P}^\beta (\sup_{\kappa \in (0,1)} \tau_{M,\kappa} \leqslant T) \rangle < \epsilon/2, \tag{5.1}$$

where $\tau_{M,\kappa} := \inf[t \geqslant 0 : |\mathbf{x}^\kappa(t)| \geqslant M]$. Let

$$A(\delta, \varrho) := [\sup_{\substack{\kappa \in (0,1),\, t, s \in [0, T] \\ 0 \leqslant t - s \leqslant \delta}} |\mathbf{x}^\kappa(t) - \mathbf{x}^\kappa(s)| > \varrho]. \tag{5.2}$$

Then, (5.1) implies that for any δ and $\varrho > 0$ we have

$$\begin{aligned}
&\langle \mathbb{P}^\beta (A(\delta, \varrho)) \rangle \\
&\quad \leqslant \langle \mathbb{P}^\beta (\tau_{M,\kappa} < T) \rangle + \langle \mathbb{P}^\beta (A(\delta, \varrho), \tau_{M,\kappa} \geqslant T) \rangle \\
&\quad \leqslant \epsilon/2 + \langle \mathbb{P}^\beta (\sqrt{2} \sup_{\substack{t, s \in [0, T] \\ 0 \leqslant t - s \leqslant \delta}} |\beta(t) - \beta(s)| + 2\,\delta \sup_{t \in [0, T], |x| \leqslant M} |V(t, x)| > \varrho) \rangle
\end{aligned}$$

and tightness follows upon an elementary application of ref. 20, p. 55, Theorem 8.2 and the estimates of the tails of the supremum of a stationary Gaussian field provided by Theorem 5.2 on p. 120 of ref. 19.

According to Skorochod's representation theorem there exist $\mathscr{X}$-valued random elements

$$\tilde{\mathscr{M}}^\kappa = (\tilde{\mathbf{x}}^\kappa(\cdot), \tilde{\eta}^\kappa(\cdot), \tilde{V}^\kappa(\cdot), \tilde{W}^\kappa(\cdot), \tilde{\beta}^\kappa(\cdot)), \qquad \kappa \in (0, 1],$$
$$\tilde{\mathscr{M}} = (\tilde{\mathbf{x}}(\cdot), \tilde{\eta}(\cdot), \tilde{V}(\cdot), \tilde{W}(\cdot), \tilde{\beta}(\cdot)),$$

such that $(\tilde{\mathscr{M}}^\kappa)_{\kappa \in (0,1)}$ and $\tilde{\mathscr{M}}$ are defined on the same probability space $\tilde{\mathfrak{B}}$, $\mathscr{L}(\mathscr{M}^\kappa) = \mathscr{L}(\tilde{\mathscr{M}}^\kappa)$, $\kappa \in (0, 1]$ and $\tilde{\mathscr{M}}^\kappa$ converges almost surely to $\tilde{\mathscr{M}}$. We will show that $(\tilde{\mathfrak{B}}, \tilde{V}(\cdot), \tilde{\mathbf{x}}(\cdot))$ is a weak solution to (1.1). First note that the laws of $(V(\cdot), W(\cdot))$ and $(\tilde{V}^\kappa(\cdot), \tilde{W}^\kappa(\cdot))$ are the same. Thus $(\tilde{V}(\cdot), \tilde{W}(\cdot))$ has the same law as $(V(\cdot), W(\cdot))$, and consequently $\tilde{V}(\cdot)$ is a stationary solution to (2.1) driven by the cylindrical Wiener process $\tilde{W}(\cdot)$. Next, passing to the limits, as $\kappa \downarrow 0$, in

$$\tilde{\eta}^\kappa(t, x) = \tilde{V}^\kappa(t, x + \tilde{\mathbf{x}}^\kappa(t)), \qquad t \geqslant 0$$

and

$$\tilde{\mathbf{x}}^\kappa(t) = \int_0^t \tilde{\eta}^\kappa(s, 0)\, \mathrm{d}s + \sqrt{2\kappa}\, \tilde{\beta}^\kappa(t), \qquad t \geqslant 0$$

we obtain the desired conclusion

$$\tilde{\mathbf{x}}(t) = \int_0^t \tilde{\eta}(s, 0)\, \mathrm{d}s = \int_0^t \tilde{V}(s, \tilde{\mathbf{x}}(s))\, \mathrm{d}s, \qquad t \geqslant 0.$$

Finally, the convergence $\mathscr{L}(\mathbf{x}^\kappa(\cdot)) \Rightarrow \mathscr{L}(\mathbf{x}(\cdot))$, as $\kappa \downarrow 0$, follows from the uniqueness of the limit. ▮

Weak Convergence of the ϵ-Regularizations. In light of the argument from the previous part it suffices only to show tightness of $\mathscr{L}_T(\mathbf{x}_\varepsilon(\cdot))$, $\varepsilon \in (0, 1]$ for any $T > 0$. Define a random variable

$$C_{T,\varepsilon} := \sup_{t \in [0,T],\, x \in \mathbb{R}^d} |V_\varepsilon(t, x)|\, \vartheta_{1/2}(x).$$

By virtue of Theorem 5.3, p. 120 of ref. 19 there exists a constant $a > 0$ such that we have

$$\sup_{\varepsilon \in (0,1]} \langle e^{aC_{T,\varepsilon}^2} \rangle < +\infty.$$

We can repeat now the argument used in the previous section to show that, with the same notation as in (5.2),

$$\lim_{\delta\downarrow 0}\ \sup_{\varepsilon\in(0,1)}\ \mathbb{P}(\mathbf{x}_\varepsilon(\cdot)\in A(\delta,\rho))=0$$

for any $\rho>0$ and tightness follows. ∎

We end this section with a pathwise existence result for the solutions of (3.6). This observation follows along the lines of the celebrated Yamada–Watanabe theorem. Assume that $2\gamma+\alpha>2$. Let $\sigma\in[0,\alpha-1)$ be such that $2\gamma+\sigma>1$. Using the same arguments as in the proof of Theorem 2.5 we obtain the existence of a weak solution $\eta_X(\cdot)$ to (3.6) starting from $X\in C^\sigma_\rho$. From Theorem 4.1 we have uniqueness of pathwise solutions to (3.6) starting from a given X. Thus a simple adaptation of the Yamada–Watanabe argument, see, e.g., Theorem 4.1.1 on p. 149 of ref. 15, leads to the following pathwise existence result for the solutions of (3.6).

Theorem 5.1. Let $2\gamma+\alpha>2$ and $\rho\in(0,\gamma/2)$. Given a filtered probability space $\mathfrak{A}$, a cylindrical Wiener process $W(\cdot)$, and $V(\cdot)$, as in Definition 2.4, there exist a $\sigma\in[0,\alpha-1)$ and an $(\mathfrak{F}_t)$-adapted $\mathscr{C}^\sigma_\rho$-valued process $\eta(\cdot)$ with continuous trajectories satisfying (3.6) with the initial condition $\eta(0)=V(0)$.

APPENDIX A

Proof of Lemma 4.2. Recall that $\gamma>0$. Let $\mathscr{C}^1$ be the space of all continuously differentiable, bounded with derivatives mappings acting from $\mathbb{R}^d$ into $\mathbb{R}^d$. The space $\mathscr{C}^1$ is equipped with the standard norm $\|\cdot\|_{\mathscr{C}^1}$. Given $\rho\geqslant 0$ we define

$$\mathscr{C}^1_\rho:=\{\psi\colon \mathbb{R}^d\to\mathbb{R}^d\colon \psi\vartheta_\rho\in\mathscr{C}^1\}.$$

We equip $\mathscr{C}^1_\rho$ with the norm $\|\psi\|_{\mathscr{C}^1_\rho}:=\|\psi\vartheta_\rho\|_{\mathscr{C}^1}$.

Let $T>0$. We will show that

$$\forall\rho\in(0,\gamma)\ \exists C\colon \sup_{t\in(0,T]}\ \|S(t)\|_{L(\mathscr{C}_\rho,\,\mathrm{C}_\rho)}\leqslant C, \tag{A.1}$$

$$\forall\rho\in(0,1/2+\gamma)\ \exists C\ \forall t\in(0,T]\colon \|\nabla S(t)\|_{L(\mathscr{C}_\rho,\,\mathrm{C}_\rho)}\leqslant Ct^{-\frac{1}{2\gamma}}, \tag{A.2}$$

and

$$\forall\rho\in(0,\gamma)\ \exists C\colon \sup_{t\in(0,T]}\ \|\nabla S(t)\|_{L(\mathscr{C}^1_\rho,\,\mathrm{C}_\rho)}\leqslant C. \tag{A.3}$$

To this end we set

$$p_\gamma(x,t)=\int_{\mathbb{R}^d} e^{-m\,|\mathbf{k}|^{2\gamma}\,t}e^{i\mathbf{k}\cdot x}\,\mathrm{d}\mathbf{k}=t^{-\frac{d}{2\gamma}}p_\gamma(xt^{-1/(2\gamma)},1) \tag{A.4}$$

and

$$q_\gamma(x,t)=\nabla p_\gamma(x,t)=i\int_{\mathbb{R}^d}\mathbf{k}e^{-m\,|\mathbf{k}|^{2\gamma}\,t}e^{i\mathbf{k}\cdot x}\,\mathrm{d}\mathbf{k}=t^{-\frac{d+1}{2\gamma}}q_\gamma(xt^{-1/(2\gamma)},1). \tag{A.5}$$

Then, by ref. 21, there are a constant $C_1<\infty$ such that $|x|^{d+2\gamma}\,|p_\gamma(x,1)|\to C_1$, as $|x|\to\infty$ and a constant $C_2<\infty$ such that $|x|^{d+2\gamma+1}\,|q_\gamma(x,1)|\to C_2$, as $|x|\to\infty$. Hence, as $p_\gamma(\cdot,1)$ and $q_\gamma(\cdot,1)$ are continuous, there is a constant $C_3<\infty$ such that

$$|p_\gamma(x,1)|\leqslant C_3\vartheta_{d/2+\gamma}(x)\quad\text{and}\quad |q_\gamma(x,1)|\leqslant C_3\vartheta_{(d+1)/2+\gamma}(x)\quad\text{for } x\in\mathbb{R}^d.$$

We have

$$\begin{aligned}|S(t)\,\psi(x)|&=\left|\int_{\mathbb{R}^d}\psi(y)\,p_\gamma(x-y,t)\,\mathrm{d}y\right|\\&\leqslant\int_{\mathbb{R}^d}|\psi(y)|\,\vartheta_\rho(y)\,|p_\gamma(x-y,t)|\,\vartheta_{-\rho}(y)\,\mathrm{d}y\\&\leqslant\|\psi\|_{\mathscr{C}_\rho}\int_{\mathbb{R}^d}|p_\gamma(x-y,t)|\,\vartheta_{-\rho}(y)\,\mathrm{d}y\\&\leqslant C_3t^{-\frac{d}{2\gamma}}\,\|\psi\|_{\mathscr{C}_\rho}\int_{\mathbb{R}^d}\vartheta_{d/2+\gamma}((x-y)\,t^{-1/(2\gamma)})\,\vartheta_{-\rho}(y)\,\mathrm{d}y.\end{aligned}$$

Thus,

$$\|S(t)\,\psi\|_{\mathrm{C}_\rho}\leqslant C_3\,I(t)\,t^{-\frac{d}{2\gamma}}\,\|\psi\|_{\mathscr{C}_\rho},$$

where

$$I(t)=\sup_{x\in\mathbb{R}^d}\vartheta_\rho(x)\int_{\mathbb{R}^d}\vartheta_{d/2+\gamma}((x-y)\,t^{-1/(2\gamma)})\,\vartheta_{-\rho}(y)\,\mathrm{d}y.$$

Similarly, we have

$$\|\nabla S(t)\,\psi\|_{\mathrm{C}_\rho}\leqslant C_3J(t)\,t^{-\frac{d+1}{2\gamma}}\,\|\psi\|_{\mathscr{C}_\rho},$$

where

$$J(t)=\sup_{x\in\mathbb{R}^d}\vartheta_\rho(x)\int_{\mathbb{R}^d}\vartheta_{(d+1)/2+\gamma}((x-y)\,t^{-1/(2\gamma)})\,\vartheta_{-\rho}(y)\,\mathrm{d}y.$$

Clearly,

$$\int_{\mathbb{R}^d}\vartheta_{d/2+\gamma}((x-y)\,t^{-1/(2\gamma)})\,\vartheta_{-\rho}(y)\,\mathrm{d}y$$

$$=t^{\frac{d}{2\gamma}}\int_{\mathbb{R}^d}\vartheta_{d/2+\gamma}(xt^{-1/(2\gamma)}-y)\,\vartheta_{-\rho}(yt^{1/(2\gamma)})\,\mathrm{d}y.$$

Thus,

$$\begin{aligned}I(t)&=t^{\frac{d}{2\gamma}}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(1+t^{1/\gamma}\,|y|^2)^\rho}{(1+|x|^2)^\rho\,(1+|xt^{-1/(2\gamma)}-y|^2)^{d/2+\gamma}}\,\mathrm{d}y\\&=t^{\frac{d}{2\gamma}}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(1+t^{1/\gamma}\,|y|^2)^\rho}{(1+t^{1/\gamma}\,|x|^2)^\rho\,(1+|x-y|^2)^{d/2+\gamma}}\,\mathrm{d}y\\&=t^{\frac{d}{2\gamma}}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(1+t^{1/\gamma}\,|x+y|^2)^\rho}{(1+t^{1/\gamma}\,|x|^2)^\rho\,(1+|y|^2)^{d/2+\gamma}}\,\mathrm{d}y.\end{aligned}$$

Using the estimate

$$(1+t^{1/\gamma}\,|x+y|^2)\leqslant 2(1+t^{1/\gamma}\,|x|^2)(1+t^{1/\gamma}\,|y|^2)$$

we conclude

$$I(t)\leqslant 2t^{\frac{d}{2\gamma}}(1+t^{1/\gamma})^\rho\int_{\mathbb{R}^d}\vartheta_{d/2+\gamma-\rho}(y)\,\mathrm{d}y.$$

Similarly, we have

$$J(t)\leqslant 2t^{\frac{d}{2\gamma}}(1+t^{1/\gamma})^\rho\int_{\mathbb{R}^d}\vartheta_{(d+1)/2+\gamma-\rho}(y)\,\mathrm{d}y,$$

and the desired conclusion follows from the integrability of $\vartheta_{d/2+\gamma-\rho}$ for $\rho<\gamma$ and $\vartheta_{(d+1)/2+\gamma-\rho}$ for $\rho<1/2+\gamma$. Estimate (A.3) follows immediately from the integration by parts formula, and from the equivalence of the norm $\|\cdot\|_{\mathscr{C}^1_\rho}$ and $|||\cdot|||_{\mathscr{C}^1_\rho}$ given by

$$|||\psi|||_{\mathscr{C}^1_\rho}:=\|\psi\|_{\mathscr{C}_\rho}+\sup_{x\in\mathbb{R}^d}|\nabla\psi(x)|_{\mathbb{R}^d\oplus\mathbb{R}^d}\,\vartheta_\rho(x),\qquad \psi\in\mathscr{C}^1_\rho.$$

Having shown (A.1)–(A.3) we complete the proof of Lemma 4.2 by showing that

$$\forall \sigma \in (0,1)\ \forall \rho \in (0,\gamma)\ \exists C\ \forall t \in (0,T]:\ \|\nabla S(t)\|_{L(\mathscr{C}^{\sigma}_{\rho},\, \mathrm{C}_{\rho})} \leqslant C t^{-\frac{1-\sigma}{2\gamma}}.$$

To derive this from (A.2) and (A.3) it is enough to show that $\mathscr{C}^{\sigma}_{\rho}$ is equal to the real-interpolation space $(\mathscr{C}_{\rho}, \mathscr{C}^{1}_{\rho})_{\sigma, +\infty}$. This follows from the well-know identity $(\mathscr{C}_0, \mathscr{C}^1_0)_{\sigma,+\infty} = \mathscr{C}^{\sigma}_{0}$, see, e.g., ref. 22, and from the fact that $\psi \to \psi \vartheta_{\rho}$ is an isomorphism between spaces $\mathscr{C}^{\sigma}_{\rho}$ and $\mathscr{C}^{\sigma}_{0}$, $\sigma \in [0,1]$. ▮

APPENDIX B

Assume that $\rho \in (d/2, d/2 + \Theta(\gamma))$. Let $\mathscr{S}(\rho)$ be the space of all $\psi \in \mathscr{S}_d$ such that $0 \notin \operatorname{supp} \widehat{\vartheta_{\rho}\psi}$. Given $\kappa \geqslant 0$ denote by S_{κ} the semigroup generated by $-A + \kappa\Delta$ on $\mathbb{L}^2$. Note that $S_{\kappa}(t) = S_0(t)\, T_{\kappa}(t)$, where T_{κ} is a C_0-semigroup generated by $\kappa \Delta$, and that S_0 and T_{κ} commute. Then, see ref. 14, there are unique extensions of S_0 and T_{κ} to C_0-semigroups S_{ρ} and $T_{\kappa,\rho}$ on $\mathbb{L}^2_{\rho}$. Hence, there is a unique extension of S_{κ} to C_0-semigroup $S_{\kappa,\rho}$ on $\mathbb{L}^2_{\rho}$, and $S_{\kappa,\rho}(t) = S_{\rho}(t)\, T_{\kappa,\rho}(t)$, $t \geqslant 0$. Let us denote by $-A_{\kappa,\rho}$ the generator of $S_{\kappa,\rho}$. Our goal is to show that $\mathscr{S}(\rho)$ is a core of $(A_{\kappa,\rho})^*$, and that

$$(A_{\kappa,\rho})^* \psi = \vartheta_{-\rho}(-A + \kappa\Delta)(\vartheta_{\rho}\psi) \qquad \text{for} \quad \psi \in \mathscr{S}(\rho). \tag{B.1}$$

First, we note that for any $\theta \in \mathbb{R}$, $\psi \mapsto \vartheta_{\theta}\psi$ is a homeomorphizm on $\mathscr{S}_d$, and that $(A + \kappa\Delta)(\vartheta_{\rho}\psi) \in \mathscr{S}_d$ for any $\psi \in \mathscr{S}(\rho)$. Moreover, $(-A + \kappa\Delta)(\vartheta_{\rho}\psi) \in \mathscr{S}(\rho)$ for any$\psi \in \mathscr{S}(\rho)$. In fact, the mapping

$$\mathscr{S}(\rho) \ni \psi \mapsto \vartheta_{-\rho}(-A + \kappa\Delta)(\vartheta_{\rho}\psi) \in \mathscr{S}(\rho)$$

is bijective.

The inclusion $\mathscr{S}(\rho) \subseteq D(A_{\kappa,\rho})^*$ and (B.1) hold since $\mathscr{S}_d$ is a core of $A_{0,\rho}$, see ref. 14, Proposition 2.iii, and since for $\varphi \in \mathscr{S}_d$, $\psi \in \mathscr{S}(\rho)$,

$$\begin{aligned}\langle A_{\kappa,\rho}\varphi, \psi\rangle_{\mathbb{L}^2_{\rho}} &= \langle (-A + \kappa\Delta)\, \varphi, \vartheta_{\rho}\psi\rangle = \langle \varphi, (-A + \kappa\Delta)(\vartheta_{\rho}\psi)\rangle \\ &= \langle \varphi, \vartheta_{-\rho}(-A + \kappa\Delta)(\vartheta_{\rho}\psi)\rangle_{\mathbb{L}^2_{\rho}}.\end{aligned}$$

Similarly, for $t \geqslant 0$ and $\psi \in \mathscr{S}(\rho)$ we have $(S_{\kappa,\rho}(t))^* \psi = \vartheta_{-\rho} S_{\kappa}(t)(\vartheta_{\rho}\psi)$. Moreover, the mappings

$$\mathscr{S}(\rho) \ni \psi \mapsto \vartheta_{-\rho} S_{\kappa}(t)(\vartheta_{\rho}\psi) \in \mathscr{S}(\rho), \qquad t \geqslant 0$$

are bijective. Summing up we have $\mathscr{S}(\rho) \subseteq D(A_{\kappa,\rho})^*$, (B.1) and for any $t \geqslant 0$, $(S_{\kappa,\rho}(t))^* (\mathscr{S}(\rho)) = \mathscr{S}(\rho)$. Since $\mathscr{S}(\rho)$ is dense in $\mathbb{L}^2_{\rho}$ and invariant with respect to $(S_{\kappa,\rho})^*$ it is a core of $(A_{\kappa,\rho})^*$, see, e.g., ref. 23, Proposition 3.3.

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REFERENCES

1. U. Frisch, *Turbulence, the Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
2. A. Yu. Veretennikov, On strong solutions of stochastic differential equations, *Theory Probab. Appl.* **24**:354–366 (1979).
3. W. E and E. Vanden Eijden, Generalized flows, intrinsic stochasticity, and turbulent transport, *Proc. Natl. Acad. Sci. USA* **97**:8200–8205 (2000).
4. W. E and E. Vanden Eijden, Turbulent Prandtl number effect on passive scalar advection, *Physica D* **152/153**:636–645 (2001).
5. K. Gawedzki and P. Horvai, Preprint available in arXiv:nlin.CD/0309027 (2003).
6. M. Avellaneda and A. J. Majda, Mathematical models with exact renormalization for turbulent transport, *Comm. Math. Phys.* **131**:381–429 (1990).
7. R. Carmona, Transport properties of Gaussian velocity fields, in *First S.M.F Winter School in Random Media, Rennes 1994, Real and Stochastic Analysis: Recent Advances*, M. M. Rao, ed. (CRC, Boca Raton, FL, 1997), pp. 9–63.
8. R. J. Adler, *Geometry of Random Fields* (Wiley, New York, 1981).
9. M. Chaves, K. Gawedzki, P. Horvai, A. Kupiainen, and M. Vergassola, Lagrangian dispersion in self-similar velocity ensembles, *J. Stat. Phys.* **113**:643–692 (2003).
10. K. Gawedzki and M. Vergassola, Phase transition in the passive scalar advection, *Physica D* **138**:63–90 (2000).
11. A. C. Fannjiang, Invariance principle for inertial-scale behavior of scalar fields in Kolmogorov type turbulence, *Physica D* **179**:161–182 (2003).
12. A. C. Fannjiang, Convergence of passive scalar fields in Ornstein–Uhlenbeck flows to Kraichnan's model, *J. Stat. Phys.* **114**:115–135 (2004).
13. G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions* (Cambridge University Press, Cambridge, 1996).
14. A. Fannjiang, T. Komorowski, and S. Peszat, Lagrangian dynamics for a passive tracer in a class of Gaussian Markovian flows, *Stochastic Processes Appl.* **97**:171–198 (2002).
15. N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes* (North-Holland, Amsterdam, 1981).
16. T. Yamada and S. Watanabe, On the uniqueness of solutions of stochastic differential equations, *J. Math. Kyoto Univ.* **11**:155–167 (1971).
17. M. Ondreját, Uniqueness for stochastic evolution equations in Banach spaces, *Dissertationes Math.*, to appear.
18. C. Stone, Weak convergence of stochastic processes defined on semi-infinite time intervals, *Proc. Amer. Math. Soc.* **14**:694–696 (1963).

19. R. J. Adler, *An Introduction to Continuity Extrema and Related Topics for General Gaussian Processes*, Info Lecture Notes 12, (Inst. Math. Stat. Hayward, 1990).
20. P. Billingsley, *Convergence of Probability Measures* (Wiley, New York, 1968).
21. R. M. Blumenthal and R. K. Getoor, Some theorems on stable processes, *Trans. Amer. Math. Soc.* **95**:263–273 (1960).
22. A. Lunardi, *Analytic Semigroup and Optimal Regularity in Parabolic Problems* (Birkhäuser, Bassel, 1995).
23. S. Ethier and T. Kurtz, *Markov Processes* (Wiley, New York, 1986).
24. S. D. Eidel'man, *Parabolic Systems* (North-Holland, Amsterdam, 1969).
25. T. Funaki, Regularity properties for stochastic partial differential equations of parabolic type, *Osaka J. Math.* **28**:495–516 (1991).
26. Y. Le Jan and O. Raimond, Integration of Brownian vector fields, *Ann. Probab.* **30**: 826–873 (2002).